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1988 J. Phys. A: Math. Gen. 21 1959

(http://iopscience.iop.org/0305-4470/21/9/012)

J. Phys. A: Math. Gen. **21** (1988) 1959-1970. Printed in the UK

Solutions of Diophantine equations and degree-one polynomial zeros of Racah coefficients

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Received 23 November 1987, in final form 20 January 1988

Abstract. It is shown that the complete set of polynomial zeros of degree one of the Racah coefficients can be obtained only from the full eight-parameter solution of the multiplicative Diophantine equation: $xyz = uvw$ subject to the constraint $z = x + y + u + v + w$. All other parametric solutions recently obtained are shown to represent only proper subsets of the complete set.

1. Introduction

Racah coefficients are related to 6-j symbols by

$$W(j_1 j_2 l_2 l_1; j_3 l_3) = (-1)^{j_1+j_2+l_1+l_2} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$$

where the 6-j symbol can be expressed in the form (Regge 1959)

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = N \sum_P (-1)^P (P+1)! \left(\prod_{i=1}^{4} (P-\alpha_i)! \prod_{k=1}^{3} (\beta_k - P)! \right)^{-1} \tag{1.1}$$

with

$$N = \Delta(j_1 j_2 j_3)\Delta(l_1 l_2 j_3)\Delta(j_1 l_2 l_3)\Delta(l_1 j_2 l_3)$$

where

$$\Delta(pqr) = \{(p+q-r)!(p-q+r)!(-p+q+r)!/(p+q+r+1)!]^{1/2}.$$

The notation is such that (Biedenharn and Louck 1981, p 430)

$$\begin{aligned} \alpha_1 &= j_1 + j_2 + j_3 & \beta_1 &= j_1 + l_1 + j_2 + l_2 \\ \alpha_2 &= l_1 + l_2 + j_3 & \beta_2 &= j_1 + l_1 + j_3 + l_3 \\ \alpha_3 &= j_1 + l_2 + l_3 & \beta_3 &= j_2 + l_2 + j_3 + l_3 \\ \alpha_4 &= l_1 + j_2 + l_3 \end{aligned} \tag{1.2}$$

with

$$\sum_{i=1}^{4} \alpha_i = \sum_{k=1}^{3} \beta_k = 2 \sum_{k=1}^{3} (j_k + l_k). \tag{1.3}$$

0305-4470/88/091959+12\$02.50

The summation in (1.1) is over those integer values of P for which

$$\max(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leqslant P \leqslant \min(\beta_1, \beta_2, \beta_3). \tag{1.4}$$

The symmetries of the 6-j symbol are made clear by rewriting it in the form of a Bargmann array (Bargmann 1962, Shelepin 1964)

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{bmatrix} \beta_1-\alpha_1 & \beta_2-\alpha_1 & \beta_3-\alpha_1 \\ \beta_1-\alpha_2 & \beta_2-\alpha_2 & \beta_3-\alpha_2 \\ \beta_1-\alpha_3 & \beta_2-\alpha_3 & \beta_3-\alpha_3 \\ \beta_1-\alpha_4 & \beta_2-\alpha_4 & \beta_3-\alpha_4 \end{bmatrix}. \tag{1.5}$$

Without loss of generality it may therefore be assumed that

$$\alpha_1 = \max(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \qquad \beta_1 = \min(\beta_1, \beta_2, \beta_3). \tag{1.6}$$

Following Brudno and Louck (1985) the weight of the 6-j symbol (1.5) is defined to be the minimum entry in the Bargman array. With the choice (1.6) the weight of the 6-j symbol is $\beta_1 - \alpha_1$ and the number of terms in its expansion (1.1) is just one more than this number.

The weight of a non-vanishing 6-j symbol is necessarily non-negative. Indeed it is well known that trivial zeros of the Racah coefficients and the corresponding 6-j symbols are associated with the vanishing of the factors $\Delta(pqr)$ in N if one or more of the usual triangle conditions $\beta_j - \alpha_i \geqslant 0$ is violated. Weight-zero 6-j symbols are always non-vanishing since their expansion consists of a single term. However, by virtue of the alternating signs in (1.1) non-trivial zeros may occur for 6-j symbols of positive weight. Recently, such non-trivial zeros, which are also known as polynomial zeros, have been the subject of much study.

Following the publication of the paper by Koozekanani and Biedenharn (1974), Biedenharn and Louck (1981, p 415) reviewed the situation and tabulated about 1400 distinct non-trivial zeros of 6-j symbols. These have been classified by Srinivasa Rao and Rajeswari (1985). Of these zeros, 33 have been accounted for through their connection with exceptional Lie algebras as discussed most recently by Van der Jeugt *et al* (1983), De Meyer and Vanden Berghe (1984) and Vanden Berghe *et al* (1984) (see also Srinivasa Rao 1985).

Using a quite different approach several papers have been published on the topic of non-trivial zeros of Racah coefficients associated with 6-j symbols of weight

$$\beta_1 - \alpha_1 = 1. \tag{1.7}$$

Such non-trivial zeros are said to be of degree one. They arise from those expansions of the form (1.1) which contain just two mutually cancelling terms of opposite sign. Srinivasa Rao and Rajeswari (1984) have pointed out that the cancellation occurs if and only if

$$(\beta_2 - \alpha_1)(\beta_3 - \alpha_1)(\beta_1 + 1) = (\beta_1 - \alpha_2)(\beta_1 - \alpha_3)(\beta_1 - \alpha_4) \tag{1.8}$$

where the convention (1.6) has been adopted, and the parameters defined by (1.2) are subject to the condition (1.3) and the constraint (1.7).

To make contact with the work of Brudno (1985) and of Brudno and Louck (1985), who independently arrived at the same result, it is convenient to change the notation of (1.5) to give

$$\begin{Bmatrix} \frac{1}{2}(x+u+v-t) & \frac{1}{2}(y+u+w-t) & \frac{1}{2}(x+y+v+w-2t) \\ \frac{1}{2}(x+w) & \frac{1}{2}(y+v) & \frac{1}{2}(x+y+u-t) \end{Bmatrix} = \begin{bmatrix} t & x & y \\ u & x+u-t & y+u-t \\ w & x+w-t & y+w-t \\ v & x+v-t & y+v-t \end{bmatrix} \tag{1.9}$$

and to introduce

$$z=\beta_1+1. \tag{1.10}$$

Term by term comparison of the Bargmann arrays in (1.5) and (1.9) defines the change of variables precisely. It follows that the weight-one restriction (1.7) corresponds to $t=1$. The condition (1.8) for degree-one polynomial zeros now takes the form of the multiplicative Diophantine equation

$$xyz=uvw. \tag{1.11}$$

By virtue of (1.7) and (1.10) the positive integers x, y, z, u, v and w in (1.11) are subject to the constraint

$$z=x+y+u+v+w. \tag{1.12}$$

After giving three one-parameter formulae for degree-one zeros, Brudno (1985) presented without proof a more general nine-parameter formula subject to one constraint. That this formula gives all possible degree-one zeros was subsequently proved by Brudno and Louck (1985) by explicitly solving the multiplicative Diophantine equation (1.11). They went on to relate (1.11) and (1.12) to a pair of Diophantine equations involving equal sums of like powers:

$$X^3+Y^3+Z^3=U^3+V^3+W^3 \tag{1.13}$$

and

$$X+Y+Z=U+V+W. \tag{1.14}$$

They located a two-parameter solution due to Gerardin (Dickson 1952, pp 565, 713). Bremner (1986) extended the investigation of (1.13) and (1.14) to produce two four-parameter solutions and related them to the Brudno and Louck solution of (1.11). Finally, Bremner and Brudno (1986) solved the same Diophantine equations to obtain another four-parameter solution which they claimed gave all degree-one zeros of the 6-j symbols.

In this paper we return to the multiplicative Diophantine equation (1.11) and establish the most general solution by invoking a theorem due to Bell (1933). This theorem is proved and applied to the problem of polynomial zeros in the following section. The various parametrisations of solutions referred to above are summarised in § 3 and attention is drawn to deficiencies in the arguments of Bremner and Brudno (1986). In particular, it is shown that their four-parameter solution, along with all the others with fewer than nine parameters, is in a certain sense incomplete. In § 4 the implementation of an algorithm for the determination of all polynomial zeros by means of a nine-parameter formula subject to one constraint and a set of greatest common divisor conditions is discussed. Finally, the claims made in § 3 are substantiated by means of explicit examples in the conclusion.

2. Bell's theorem

Bell (1933) studied seven types of multiplicative Diophantine equations and by making use of 'reciprocal arrays' established the most general solutions expressed in terms of the minimum possible number of parameters. The solutions all involve greatest common number of parameters. The solutions all involve greatest common divisor (GCD) conditions. As usual, given any non-negative integers x and y the GCD of x and y is denoted by (x, y). We write $x|y$ if x divides y in the sense that y/x is an integer, and conversely write $x\nmid y$ if x does not divide y. From our point of view the key theorem is the following.

Bell's theorem. Every solution of the multiplicative Diophantine equation

$$x_1x_2\ldots x_n = u_1u_2\ldots u_n \tag{2.1}$$

can be expressed in the form

$$x_i = \prod_{j=1}^{n} \phi_{ij} \qquad u_j = \prod_{i=1}^{n} \phi_{ij} \qquad \text{for } i, j = 1, 2, \ldots, n \tag{2.2}$$

where the n^2 independent parameters ϕ_{ij} with $i, j = 1, 2, \ldots, n$ are positive integers which can be arranged in an $n\times n$ array $A(\phi)$ with ϕ_{ij} at the intersection of the ith row and jth column subject to the GCD conditions

$$(x_i, u_i) = \phi_{ii} \qquad \text{for } i = 1, 2, \ldots, n \tag{2.3}$$

applying to the diagonal elements of the array.

It is to be noted that this statement of Bell's theorem only differs from the original in that the two reciprocal arrays of Bell (1933) have been replaced by a single array $A(\phi)$. To prove the theorem for all values of n it is probably simplest to follow an inductive argument reminiscent of that used by Brudno and Louck (1985) but taking into account the GCD conditions and not allowing permutations of the components x_i and u_i for $i = 1, 2, \ldots, n$. We let

$$x_1x_2\ldots x_n = u_1u_2\ldots u_n = N \tag{2.4}$$

and the induction argument is made with respect to the parameter N, keeping n fixed throughout.

Bell's theorem is obviously true for $N = 1$. The only solution is $x_i = u_i = 1$ for $i = 1, 2, \ldots, n$ and correspondingly $\phi_{ij} = 1$ for all $i, j = 1, 2, \ldots, n$. For the induction hypothesis we assume that all the solutions of (2.4) are given by Bell's theorem for $N = 1, 2, \ldots, M-1$ with $M > 1$.

Now we consider two cases: firstly any solution of (2.4) with $N = M$ for which

$$(x_i, u_i) = q \qquad \text{with } 1 < q \leqslant N \text{ for some } i \in \{1, 2, \ldots, n\}. \tag{2.5}$$

Cancelling q throughout (2.4) with $N = M$ gives an equation of the same type with $N = M/q$. By the induction hypothesis all the solutions of this equation are given by Bell's theorem for some array $A(\phi')$. Having divided both x_i and u_i by q it is clear that $\phi'_{ii} = 1$. Simply multiplying this element at the intersection of the ith row and ith column of the array $A(\phi')$ by q and leaving all the other elements unaltered then gives the required array $A(\phi)$ for the original solution of (2.4) with $N=M$. The GCD conditions are automatically satisfied.

Secondly, there remains only the case for which

$$(x_i, u_i)=q=1 \qquad \text{for all } i\in\{1,2,\ldots,n\}. \tag{2.6}$$

Since $M>1$ it follows that there exists some prime $p>1$ such that $p|M$. Correspondingly there exists x_i and u_j with $i\neq j$ such that $p|x_i$ and $p|u_j$. Cancelling p throughout (2.4) with $N=M$ then gives an equation of the type (2.4) with $N=M/p$. By the induction hypothesis any solution of this equation gives an array $A(\phi')$ satisfying the GCD conditions. In fact by virtue of (2.6) all the diagonal entries are 1. Multiplying the entry ϕ'_{ij} at the intersection of the ith row and jth column by p and again leaving all the other elements unaltered then gives the array $A(\phi)$ required to represent the solution of (2.4) with $N=M$. The GCD condition is still satisfied because the diagonal entries are still just 1.

This completes the induction argument and Bell's theorem is proved provided that we can show that the n^2 parameters are genuinely independent. This can be seen most easily by considering those solutions of (2.1) of the form (2.2) for which the n^2 parameters ϕ_{ij} take on n^2 distinct prime values. To generate the complete set of such solutions for arbitrary N it is clear that all n^2 parameters are required.

It is worth pointing out that in general for $n>3$ it is not true that all distinct arrays $A(\phi)$ satisfying the GCD conditions (2.3) give distinct solutions. However, this is the case for $n\leqslant 3$. This is trivial for $n=1$ and $n=2$. For $n=3$ it can be proved by noting that if $A(\phi)$ and $A(\phi')$ are different but correspond to the same solution of (2.1), then there exists some prime $p>1$ and some pair (i,j) with $i\neq j$ such that

$$p|\phi_{ij} \qquad p\nmid\phi'_{ij}. \tag{2.7}$$

In order that the arrays $A(\phi)$ and $A(\phi')$ correspond to the same solution the products of the elements in their ith rows must coincide, as must the products of the elements in their jth columns. Hence, taking into account the fact that their diagonal elements also coincide, there must exist k such that

$$p|\phi'_{ik} \qquad \text{with } \{i,j,k\}\subseteq\{1,2,\ldots,n\} \text{ and } k\neq i\neq j\neq k \tag{2.8}$$

and m such that

$$p|\phi'_{mj} \qquad \text{with } \{i,j,m\}\subseteq\{1,2,\ldots,n\} \text{ and } m\neq i\neq j\neq m. \tag{2.9}$$

It follows that if $n=3$ then $k=m$. Hence

$$(x_k, u_k)=\mu p\phi'_{kk}=\mu p(x_k, u_k) \tag{2.10}$$

for some integer $\mu\geqslant 1$, and we have a contradiction for $p>1$. It follows that, for $n=3$, distinct arrays $A(\phi)$ satisfying the GCD conditions (2.3) lead by means of (2.2) to distinct solutions (2.1) and vice versa. Applying this result to the degree-one zeros discussed in the introduction we obtain the following.

Theorem. The degree-one polynomial zeros of 6-j symbols are all given, up to symmetry transformations, by (1.9) with $t=1$ and $xyz=uvw$, where $z=x+y+u+v+w$. All possible solutions to these equations are specified by the distinct arrays of the form

$$\begin{array}{c|ccc} & u & v & w \\ \hline x & a & b & c \\ y & d & e & f \\ z & g & h & i \end{array} \tag{2.11}$$

where (x, y, z, u, v, w) are given by the products of the elements in the appropriate rows and columns of this array. The entries $(a, b, c, d, e, f, g, h, i)$ take on all positive integer values consistent with the conditions

$$ghi = adg + beh + cfi + abc + def \tag{2.12}$$

and

$$\begin{aligned}(b, d) = (b, g) = (b, f) = (c, d) = (c, g) = (c, h)\\ = (d, h) = (f, g) = (f, h) = 1.\end{aligned} \tag{2.13}$$

This theorem makes it obvious that the complete set of solutions of the equation (2.4) constrained by (2.3) requires a minimum of eight parameters (since (2.12) can be used to eliminate one of the nine parameters). Brudno (1985) has written down the nine-parameter solution of (2.4) for $n = 3$ and in terms of this he showed that the polynomial zeros of degree one of the Racah coefficient are given by

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(x+u+v-1) & \frac{1}{2}(y+u+w-1) & \frac{1}{2}(z-u-2) \\ \frac{1}{2}(x+w) & \frac{1}{2}(y+v) & \frac{1}{2}(x+y+u-1) \end{Bmatrix} \tag{2.14}$$

where x, y, z and u, v, w are given by the products of the row and column elements of the 3×3 array (2.11), respectively.

For an alternative proof of Bell's theorem (2.1) for $n = 2$ and 3, and the general result for arbitrary n by induction, refer to Srinivasa Rao *et al* (1987).

3. Comparison with other parametrisations

Brudno (1985) found the following one-parameter formulae for degree-one zeros of the 6-j symbols:

$$\begin{Bmatrix} n+2 & n+1 & 2 \\ n & n+1 & n+1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} m+2 & m+1 & m+1 \\ m & 2 & m+1 \end{Bmatrix} \tag{3.1}$$

$$\begin{Bmatrix} \frac{3}{2}x+2 & \frac{3}{2}x+2 & x+2 \\ x+\frac{3}{2} & \frac{3}{2} & \frac{3}{2}x+\frac{3}{2} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \frac{3}{2}n+\frac{1}{2} & \frac{3}{2}n+\frac{1}{2} & n+1 \\ n+\frac{1}{2} & \frac{3}{2} & \frac{3}{2}n \end{Bmatrix} \tag{3.2}$$

and

$$\begin{Bmatrix} J & 4J-1 & 3J \\ 2J+\frac{3}{2} & J+\frac{1}{2} & 2J-\frac{1}{2} \end{Bmatrix} \Rightarrow \begin{Bmatrix} b+\frac{5}{2} & 2b+1 & b+\frac{1}{2} \\ \frac{1}{2}b+\frac{1}{2} & \frac{1}{2}b+1 & \frac{3}{2}b+\frac{3}{2} \end{Bmatrix} \tag{3.3}$$

with m, n, and $b = 1, 2, \ldots,$ where in (3.1) and (3.3) the symmetries of the 6-j symbol have been exploited to write them in the form (1.9) with $t = 1$. These three cases are covered in the notation of (2.11) by means of the arrays

$$\begin{array}{c|ccc} & u & v & w \\ \hline x & 1 & 1 & m \\ y & 1 & 1 & 1 \\ z & m+2 & 3 & 1 \end{array} \qquad \begin{array}{c|ccc} & u & v & w \\ \hline x & n & 1 & 1 \\ y & 1 & 1 & 1 \\ z & 2 & 2 & n+1 \end{array} \qquad \begin{array}{c|ccc} & u & v & w \\ \hline x & 1 & 1 & 1 \\ y & 1 & 1 & b \\ z & 2b+3 & 2 & 1 \end{array}$$

respectively.

By solving the pair of Diophantine equations (1.13) and (1.14) for sums of like powers, Brudno and Louck (1985) determined the two-parameter solution specified by the array

$$\begin{array}{c|ccc} & u & v & w \\ \hline x & q & 3q-2p & 1 \\ y & \frac{1}{2}p & 1 & q-p \\ z & 3 & q-\frac{1}{2}p & 3q-p \end{array} \Rightarrow \begin{array}{c|ccc} & u & v & w \\ \hline x & 2b+h & 2b+3h & 1 \\ y & b & 1 & h \\ z & 3 & b+h & 4b+3h \end{array} \tag{3.4}$$

with $b, h = 1, 2, \ldots$. By the same means Bremner (1986) obtained the solutions given by the arrays

$$\begin{array}{c|ccc} & u & v & w \\ \hline x & \gamma-\delta & \beta & 2 \\ y & \alpha-2\beta & 1 & \delta \\ z & 1 & \gamma+2\delta & \alpha+\beta \end{array} \Rightarrow \begin{array}{c|ccc} & u & v & w \\ \hline x & a & d & 2 \\ y & b & 1 & h \\ z & 1 & a+3h & b+3d \end{array} \tag{3.5}$$

with $a, b, d, h = 1, 2, \ldots$ and

$$\begin{array}{c|ccc} & u & v & w \\ \hline x & \alpha\gamma-4\alpha\delta+\beta\gamma-\beta\delta & \frac{1}{2}(-\gamma+5\delta) & \alpha \\ y & \frac{1}{2}(\alpha\gamma-5\alpha\delta+\beta\gamma-2\beta\delta) & 1 & -\alpha\gamma+5\alpha\delta-\beta\delta \\ z & 1 & -\alpha^2\gamma+5\alpha^2\delta+\beta^2\gamma-2\beta^2\delta & \frac{3}{2}\delta \end{array}$$

$$\Rightarrow \begin{array}{c|ccc} & u & v & w \\ \hline x & ps+3qr+4qs & \frac{1}{2}r & p \\ y & -\frac{1}{2}pr+qr+\frac{3}{2}qs & 1 & pr-qr-qs \\ z & 1 & p^2r+2q^2r+3q^2s & \frac{3}{2}r+\frac{3}{2}s \end{array} \tag{3.6}$$

with $p, q, r, s = 1, 2, \ldots$ subject to the constraint $q(2r+3s) > pr > q(r+s)$. The culmination of this approach is the four-parameter formula of Bremner and Brudno (1986) which purports to give the complete solution to the problem. However it is not difficult to see that the array corresponding to their solution (Bremner and Brudno 1986, equation (27)) can be written in the form

$$\begin{array}{c|ccc} & u & v & w \\ \hline x & r & 1 & pq-rs \\ y & s & pq-rs & 1 \\ z & p+q+r+s & p & q \end{array} \Rightarrow \begin{array}{c|ccc} & u & v & w \\ \hline x & a & 1 & fi-ab \\ y & b & fi-ab & 1 \\ z & a+b+f+i & f & i \end{array} . \tag{3.7}$$

Clearly this solution is not complete in the sense that varying p, q, r and s over all positive integers subject to the obvious requirement $pq > rs$ does not generate all possible solutions. For example, the very well known solution

$$\begin{Bmatrix} 2 & 2 & 2 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{Bmatrix} = 0 \tag{3.8}$$

corresponding to $x = y = 1$, $u = v = w = 2$ and $z = 8$, specified by the array

$$\begin{array}{c|ccc} & u & v & w \\ \hline x & 1 & 1 & 1 \\ y & 1 & 1 & 1 \\ z & 2 & 2 & 2 \end{array} \tag{3.9}$$

cannot be obtained from (3.7) with integer parameters. A clue to this omission comes from noticing that it may be recovered from (3.7) by setting

$$p = q = 2(\tfrac{1}{3})^{1/3} \qquad r = s = (\tfrac{1}{3})^{1/3}. \tag{3.10}$$

The explanation for this lies in the fact that in deriving the solution (3.7) to (1.13) and (1.14) Bremner and Brudno have made successive transformations from the parameters (X, Y, Z, U, V, W) to $(\alpha, \beta, \gamma, \delta)$ to (p, q, r, s). However, at one stage a denominator is removed, with the justification that their original equations (1.13) and (1.14) are homogeneous. Quite apart from the fact that such a step is not appropriate in dealing with Diophantine equations, the weight-one 6-j symbols are not themselves homogeneous in any of the sets of parameters since their definition involves setting $t = 1$ in (1.9). In terms of the parameters (p, q, r, s) of Bremner and Brudno (1986), the hidden change of parameters is such that in their solution (27) these four parameters should be replaced by

$$\begin{aligned} p' &= p/[2(pq - rs)]^{1/3} \qquad & q' &= q/[2(pq - rs)]^{1/3} \\ r' &= r/[2(pq - rs)]^{1/3} \qquad & s' &= s/[2(pq - rs)]^{1/3}. \end{aligned} \tag{3.11}$$

The substitution of the values $p = q = 4$ and $r = s = 2$ then gives $p' = q' = 2(\frac{1}{3})^{1/3}$ and $r' = s' = (\frac{1}{3})^{1/3}$, which as noted in (3.10) are precisely those values enabling (3.9) to be recovered from (3.7).

It is perhaps worth pointing out that a four-parameter solution very closely related to (3.7) may be very trivially obtained from the complete solution (2.11) merely by rearranging the elements as below, taking care to preserve all row and column products:

$$\begin{array}{c|ccc} & u & v & w \\ \hline x & abc & 1 & 1 \\ y & def & 1 & 1 \\ z & g/bcef & beh & cfi \end{array} \Rightarrow \begin{array}{c|ccc} & u & v & w \\ \hline x & a & 1 & 1 \\ y & d & 1 & 1 \\ z & \dfrac{a+d+h+i}{hi-ad} & h & i \end{array}. \tag{3.12}$$

This is a four-parameter formula for the complete solution in which the parameters a, d, h, i are positive integers. This same complete solution to the Diophantine equations (1.11) and (1.12) can be obtained even more trivially by setting $x = a$, $y = d$, $v = h$ and $w = i$. Solving for z and u then gives

$$z = (a + d + h + i)hi/(hi - ad) \qquad u = adz/hi \tag{3.13}$$

precisely as indicated in (3.12). For each set of such parameters it is necessary to check that z is a positive integer, which would ensure u being an integer. In terms of

Bell's theorem (2.11), since $b=c=e=f=1$, we have, in fact, a five-parameter solution, which reduces to a four-parameter solution due to the constraint equation (2.12). Note that the fifth parameter $(a+d+h+i)/(hi-ad)$ will not always be an integer, e.g.

$$\begin{Bmatrix} 7 & \frac{9}{2} & \frac{9}{2} \\ \frac{5}{2} & 4 & 4 \end{Bmatrix} = \begin{Bmatrix} \frac{9}{2} & \frac{9}{2} & 7 \\ 4 & 4 & \frac{5}{2} \end{Bmatrix}$$

has the four-parameter solution

$$x=a=2 \qquad y=d=2 \qquad v=h=6 \qquad w=i=6$$

so that the fifth parameter

$$(a+d+h+i)/(hi-ad)=\tfrac{1}{2}$$

which is the exceptional case referred to in table 2.

4. Algorithms

An algorithm for finding all the solutions of (2.12) was given by Srinivasa Rao and Rajeswari (1986) wherein (2.12) was reduced to the quadratic Diophantine equation

$$\alpha xy=\beta x+\gamma y+\delta \tag{4.1}$$

with $\alpha=i$, $x=g$, $y=h$, $\beta=ad$, $\gamma=be$ and $\delta=abc+def+cfi$, the solutions of which were given by Brahmagupta (cf Dickson 1952, p 64). Here we present an alternate algorithm to reduce (2.12) to the linear Diophantine equation of the form

$$\alpha x+\beta y=\gamma \tag{4.2}$$

which is widely discussed in the literature (cf Dickson 1952).

Since the nine integer parameters in the array (2.11) can take non-zero integer values we consider the following.

(i) Let seven of these take successive values 1 to 10 (say) and these are arranged into a nest of loops. The two parameters excluded from this nest should belong to independent rows and columns, e.g. (a, e), (e, i), (b, d), (c, e), (e, g), etc, in (2.11).

(ii) The nine relative prime conditions to be satisfied by the parameters, given by (2.13), are checked. (Note that these conditions are the direct consequence of the three GCD conditions.)

(iii) The constraint equation (2.12) now reduces to the form (4.2). For instance, if $x=a$, $y=e$, then

$$\alpha=bc+dg \qquad \beta=df+bh \qquad \gamma=i(gh-cf). \tag{4.3}$$

(iv) Solutions of (4.2) and (4.3) are sought such that $gh>cf$ and $i(gh-cf)\geqslant b(c+h)+d(f+g)$. Paoli (cf Dickson 1952, p 401) noted that if (4.2) has integral solutions, any common factor of α and β must divide γ and hence can be removed from every term. Hence, let α and β be relatively prime and positive. Let ε denote the least positive integer such that $\gamma-\alpha\varepsilon$ is divisible by β. Then every solution is given by

$$x=\varepsilon+\beta m \qquad y=(\gamma-\alpha\varepsilon)/\beta-\alpha m \tag{4.4}$$

where the values of m making x and y positive are $0, 1, 2, \ldots, E$; E being the largest integer less than $(\gamma-\alpha\varepsilon)/\alpha\beta$.

Thus all the parameters subject to the constraint (4.2) are determined.

Alternatively, a simpler algorithm is the one which arises due to the four-parameter solution given in (3.12):

(i) let all the four integer parameters, a, d, h, i take successive values 1 to 10 (say) and these are arranged into a nest of loops;

(ii) check for $hi > ad$, compute z given by (3.13), and

(iii) check for z being an integer and compute u given in (3.13).

Having obtained the values of $x = a$, $y = d$, $v = h$ and $w = i$, as well as the values of u and z, the required degree-one polynomial zero is given by (2.14).

5. Conclusions

The one-, two- and four-parametric solutions given by Brudno (3.3), Brudno and Louck (3.4), Bremner (3.5) and (3.6), and ourselves (3.12) do not yield all the polynomial zeros of degree one of the Racah coefficient. To illustrate this explicitly, first we give in table 1 the minimum values of the parameters allowed in the one-, two-, four- and eight-parametric solutions given by different authors and the corresponding arguments of

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = 0$$

with the value of the invariant

$$I = 2 \sum_{k=1}^{3} (j_k + l_k) = 3z + x + y - 5.$$

Note that

$$\begin{Bmatrix} 2 & 2 & 2 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{Bmatrix} = 0$$

corresponding to $I = 21$ is the first polynomial zero of degree one given in table III of Srinivasa Rao and Rajeswari (1985), which lists 1174 of the Regge equivalent zeros with $21 \leqslant I \leqslant 177$.

Table 1. Parametric solutions of the polynomial zeros of degree one of the Racah coefficient.

Serial number	Reference	Equation	Parameters: General	Parameters: Minimum values	Racah coefficient: j_1	j_2	j_3	l_1	l_2	l_3	Invariant I
1	Brudno (1985)	(3.1)	m	1	3	2	2	1	2	2	24
2	Brudno (1985)	(3.2)	n	1	2	2	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	21
3	Brudno (1985)	(3.3)	b	1	$\frac{7}{2}$	3	$\frac{3}{2}$	1	$\frac{3}{2}$	3	27
4†	Brudno and Louck (1985)	(3.4)	(b, h)	(1, 1)	$\frac{33}{2}$	$\frac{31}{2}$	8	11	12	$\frac{11}{2}$	137
5†	Bremner (1986)	(3.5)	(a, b, d, h)	(1, 1, 1, 1)	$\frac{13}{2}$	5	$\frac{5}{2}$	$\frac{3}{2}$	3	$\frac{9}{2}$	46
6†	Bremner (1986)	(3.6)	(p, q, r, s)	(3, 1, 1, 1)	$\frac{31}{2}$	15	$\frac{19}{2}$	12	$\frac{25}{2}$	4	137
7†	Present	(3.12)	(a, d, h, i)	(1, 1, 2, 2)	2	2	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	21
8†	Present	(2.24)	$(a, b, c, d, e, f, g, h, i)$	(1, 1, 1, 1, 1, 1, 2, 2, 2)	2	2	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	21

† Given the nine parameters of the 3×3 array, x, y, z and u, v, w are known (being the products of the row and column elements of the array, respectively). These values are used in (2.14) to obtain the arguments of the Racah coefficient.

Though, like the eight-parameter solution (2.11) and (2.12), the one- (3.2) and four- (3.12) parameter formulae also give rise to the first of the 'non-trivial' degree-one zeros, unlike the eight-parameter case, (3.2), as well as the two other one-parameter formulae (3.1) and (3.3), the two-parameter formula (3.4) and the four-parameter formulae (3.5) and (3.6), cannot generate the complete list of polynomial zeros of degree one. We illustrate this by listing in table 2 the first fifteen Regge inequivalent polynomial zeros of degree one and indicate which of the parametric solutions given in table 1 can account for them and which cannot.

Table 2. Parametric formulae and the first fifteen of the inequivalent polynomial zeros of degree one of the Racah coefficient $\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$. √ indicates that the parametric formula accounts for the zero and × that it does not. * exceptional case, see text for details.

						Serial number of parametric solutions given in table 1 (number of parameters)							
j_1	j_2	j_3	l_1	l_2	l_3	1 (1)	2 (1)	3 (1)	4 (2)	5 (4)	6 (4)	7 (4)	8 (8/9)
2	2	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	×	√	×	×	×	×	√	√
3	2	2	1	2	2	√	×	×	×	×	×	√	√
$\frac{7}{2}$	3	$\frac{3}{2}$	1	$\frac{3}{2}$	3	×	×	√	×	×	×	√	√
$\frac{7}{2}$	$\frac{7}{2}$	3	$\frac{5}{2}$	$\frac{3}{2}$	3	√	√	×	×	×	×	√	√
5	4	2	3	4	4	√	×	×	×	×	×	√	√
5	$\frac{9}{2}$	$\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{9}{2}$	×	×	√	×	×	×	√	√
5	$\frac{9}{2}$	$\frac{9}{2}$	$\frac{7}{2}$	3	3	×	×	×	×	×	×	√	√
5	5	4	$\frac{7}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	×	√	×	×	×	×	√	√
$\frac{11}{2}$	4	$\frac{7}{2}$	1	$\frac{7}{2}$	4	×	×	×	×	×	×	√	√
6	5	3	1	3	5	×	×	×	×	√	×	√	√
6	5	5	4	5	2	√	×	×	×	×	×	√	√
6	6	4	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{11}{2}$	×	×	×	×	×	×	√	√
$\frac{13}{2}$	6	$\frac{7}{2}$	3	$\frac{3}{2}$	6	×	×	√	×	×	×	√	√
7	$\frac{9}{2}$	$\frac{9}{2}$	$\frac{5}{2}$	4	4	×	×	×	×	×	×	*	√
$\frac{15}{2}$	7	$\frac{7}{2}$	2	$\frac{7}{2}$	6	×	×	×	×	√	×	√	√

Thus, from table 2, it follows that the various one-, two- and four-parametric solutions, listed as 1–7 in table 1, yield only different subsets of all the possible polynomial zeros of degree one, since according to the theorem stated herein eight parameters are necessary and sufficient to account for all the solutions of the constrained multiplicative Diophantine equation (1.11) and (1.12). However, the exceptional four-parameter solution trivially obtained by us in (3.12) gives a complete set of solutions provided, as pointed out earlier, in terms of (2.11), the fifth parameter is permitted to take non-integral values, just as the substitution of (3.10) in (3.7) allows the first of the tabulated polynomial zeros of degree one.

Acknowledgments

One of us (KSR) wishes to thank Professor T S Santhanam and Dr R Balasubramanian for discussions on the paper of E T Bell on 'Reciprocal Arrays and Diophantine Analysis', and Drs R Simon and R Jagannathan for discussions on the algorithmic solutions. He is also grateful to the Alexander von Humboldt Foundation, West Germany, for the gift of an IBM-PC/AT computer on which the numerical verification of the algorithms was carried out. Another one of us (VR) thanks the Council of Scientific and Industrial Research, Government of India, for the award of a Senior Research Fellowship.

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